Selective Feedback Stabilization of Ginzburg–Landau Spiral Waves in Circular and Spherical Geometries

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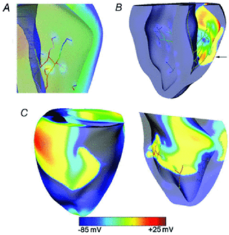
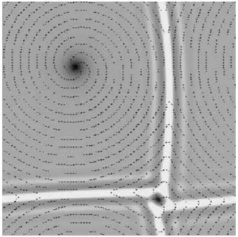
Research Focus

Explain, **predict**, and **control** formation of spiral waves.



physiology

chemistry



physics

Complex Ginzburg–Landau Equation

$$\partial_t \Psi = (1 + i\,\eta)\,\Delta_{\mathcal{M}} \Psi + \lambda\,(1 - |\Psi|^2 - i\,\beta\,|\Psi|^2)\,\Psi.$$

- $\Delta_{\mathcal{M}}$: the Laplace–Beltrami operator on $\mathcal{M}$.
- Geometry: $\mathcal{M} = B^2$ with Robin boundary, or $\mathcal{M} = S^2$.
- $\eta, \beta \in \mathbb{R}$: given parameters.
- $\lambda > 0$: adjustable bifurcation parameter.
- gauge equivariance: Ψ is a solution if and only if $e^{i\omega}\Psi$ is a solution for each fixed $\omega \in S^1$.

Ginzburg–Landau Spirals

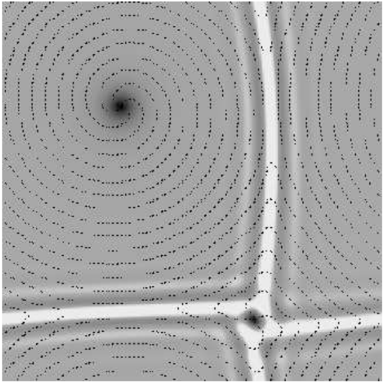


Figure: The amplitude $|\Psi|$ and level sets of the phase $\arg(\Psi) = 0, \pi$. The tip is a phase singularity; see [Aranson, Kramer, 2002].

Goal and Trilogy

Goal: Prove numerical or experimental evidence on spiral waves.

Trilogy of Research

- Part 1: **Existence**

  Analysis: Global bifurcation by symmetry breaking.

- Part 2: **Stability Analysis**

  Significance: Stable spirals are observable.

How about unstable spirals?

- Part 3: **Delayed Feedback Stabilization**

Spiral Ansatz

Consider $\mathcal{M} = S^2$ for illustration. In polar coordinates

$$(s, \varphi) \quad \mapsto \quad (\sin(\vartheta)\cos(\varphi), \sin(\vartheta)\sin(\varphi), \cos(\vartheta)),$$

decompose $L^2(S^2)$ into invariant subspaces as

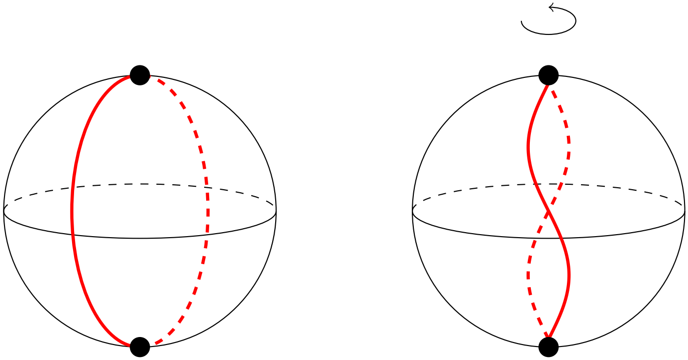
$$L^2(S^2) = \bigoplus L^2_n, \quad L^2_n := \{\psi(\vartheta, \varphi) = u(\vartheta)\, e^{in\varphi}\}.$$

Task: Fix $m \in \mathbb{N}$ and seek nontrivial solutions of the spiral Ansatz

$$\Psi(t, \vartheta, \varphi) = e^{-i\Omega t}\, u(\vartheta)\, e^{im\varphi}.$$

- Unknowns: rotation frequency $\Omega \in \mathbb{R}$ and profile $u(\vartheta)$.
- m is the number of arms; Tips are the two poles of S^2.

1-armed Spirals



Existence of Spirals

Substituting the spiral Ansatz $\Psi(t,\vartheta,\varphi) = e^{-i\Omega t}\,\psi(\vartheta,\varphi)$ yields the (elliptic) spiral wave equation:

$$(1+i\,\eta)\,\Delta_m\psi + i\,\Omega\,\psi + \lambda\,(1-|\psi|^2 - i\,\beta\,|\psi|^2)\,\psi = 0,$$

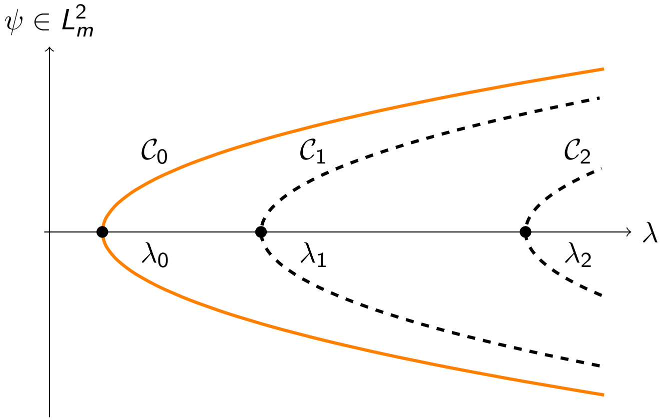
where Δ_m is the restriction of Δ_{S^2} on L^2_m.

$$\textit{Known}: \mathrm{Spec}(-\Delta_m) = \{\lambda_k := (m+k)(m+k+1) : k \in \mathbb{N}_0\}.$$

Theorem (Existence, [Dai, Lappicy, 2021])

Each $\lambda \in (\lambda_k, \lambda_{k+1}]$ admits an $\epsilon > 0$ so that the Ginzburg–Landau equation has $2k+2$ spiral waves for each $\eta, \beta \in (-\epsilon, \epsilon)$ and $\eta \neq \beta$.

Bifurcation Diagram



L^2-instability of Spirals

Local stability of spiral waves for $0 \leq |\eta|, |\beta| \ll 1$ can be reduced to the variational case $(\eta, \beta) = (0, 0)$.

- $\psi_0 \in \mathcal{C}_0$ is (locally asymptotically) stable if the principle eigenvalue μ^* of the associated linearized operator

$$\mathcal{L}[V] := \Delta_{S^2} V + \lambda \left((1 - 2|\psi_0|^2) V - |\psi_0|^2 e^{2im\varphi} \overline{V} \right)$$

  is negative.

Theorem ([Cheng, 2020])

$\mu^* \geq 0$, i.e., ψ_0 cannot be stable.

Control Triple Method

Question: Can ψ_0 be stabilized in a noninvasive way?

Answer: Yes, by introducing the delayed feedback control:

$$\partial_t \Psi = \Delta_{S^2} \Psi + \lambda (1 - |\Psi|^2) \Psi + \boxed{b \left(\Psi - h \, \Psi(t - \tau, \vartheta, \varphi - \xi) \right)}.$$

The control triple (h, τ, ξ) consists of

- **output signal** $h \in \mathbb{C}$ and $|h| = 1$;
- **time delay** $\tau > 0$;
- **space delay** $\xi \in S^1$.

Noninvasiveness

The control triple (h, τ, ξ) is called noninvasive if ψ_0 is also a solution of the control system, i.e., the control term

$$b\,(\Psi - h\,\Psi(t-\tau, \vartheta, \varphi - \xi))$$

vanishes at $\Psi(t, \vartheta, \varphi) = \psi_0(\vartheta, \varphi) = u_0(\vartheta)\, e^{im\varphi}$.

- The symmetry $u_0(\vartheta) = u_0(\pi - \vartheta)$ yields the noninvasive control triples $(h, \tau, \xi) = (e^{im\xi}, \tau, \xi)$.
- **Task**: Find noninvasive control triples and $b < 0$ so that ψ_0 becomes stable.

Step 1: Fourier Decomposition

When $\tau = 0$, the decomposition $L^2(S^2) = \bigoplus L_n^2$ yields linearized operators indexed by $n \in \mathbb{Z}$:

$$\mathcal{L}_n + b\left(1 - e^{im\xi} e^{-in\xi}\right)\mathcal{I},$$

where $\mathcal{L}_n$ is the restriction of $\mathcal{L}$ to L_n^2.

- The principal eigenvalue of $\mathcal{L}_n$ is smaller than μ_*.
- The control term shifts the spectrum of $\mathcal{L}_n$ and does not produce instability when $b < 0$.

Step 2: Spectral Gap

$$\mathcal{L}_n + b\left(1 - e^{im\xi} e^{-in\xi}\right)\mathcal{I},$$

- The nonresonance cases $n \neq m$ yield a spectral gap and so $\xi \in S^1$ and $b < 0$ exist so that ψ_0 is stabilized in L^2_n.
- For the resonance case $n = m$ the control term disappears, but it is known that ψ_0 is stable on L^2_m.
- The spectral gap persists for $0 < \tau \ll 1$, by carefully studying the characteristic equations.

Summary

We successfully apply the control triple method to stabilize spiral waves selectively, i.e., only the spiral with prescribed number of arms and symmetry is stabilized:

WHAT'S NEXT

- Can we stabilize spirals on $\mathcal{C}_j$ for $j \geq 2$?
- Can we stabilize traveling waves?